Parabolic subgroups
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Fix $k$ a field, $G/k$ affine algebraic group (e.g. $G = \text{GL}_{n,k}$, $\text{Sp}_{n,k}$, $\text{O}_{n,k}$ . . . )

**Definition**
A split torus $T$ of $G$ is subgroup of $G$ isomorphic to $\mathbb{G}_m^n$ torus is $T \subset G$ such that $T\bar{k}$ is a split torus.

- The diagonal subgroup $D_{n,k}$ is a maximal torus in $\text{GL}_{n,k}$;
- The maximal torus in $\text{Sp}_{n,k}$ is isomorphic to $\mathbb{G}_m^n$ under $(a_1, \ldots a_n) \mapsto \text{diag}(a_1, \ldots, a_n, a_1^{-1}, \ldots, a_n^{-1})$;
- If $F$ number field, $Res_{F/\mathbb{Q}}\mathbb{G}_m$ is a non-split torus.

For a torus $T$, we let

$$X(T) = \text{Hom}(T, \mathbb{G}_m)$$

be the space of characters of $T$. It’s a finite free $\mathbb{Z}$-module.
Lie algebras and Adjoint representation

The Lie algebra $\mathfrak{g}$ of $G$ is the tangent space of $G$ at the identity: $\mathfrak{g} = T_e G$. It is an algebra with the product given by the Lie bracket $[X, Y] = XY - YX$ for $X, Y \in \mathfrak{g}$.

The action of $G$ on itself by inner automorphism, induced an automorphism of the tangent space at the identity

$$\text{Ad}: G \rightarrow \text{GL}(\mathfrak{g})$$

Assume $k = \overline{k}$, $T \subset G$ a maximal torus, $G$ reductive. The vector space $\mathfrak{g}$ as a representation of $T$ decomposes as

$$\mathfrak{g} = \mathfrak{t} \oplus \bigoplus_{\alpha \in \Phi} \mathfrak{g}_\alpha$$

where $\mathfrak{t}$ is the Lie algebra of $T$. 
Example: root system of $\text{SL}_{n+1}$

The Lie algebra $\mathfrak{sl}_{n+1}$ is given by

$$\{M \in M_{(n+1) \times (n+1)} \mid \text{Tr}(M) = 0\}$$

Let $E_{i,j}$ be the matrix with 1 in the $(i, j)$ entry and 0 elsewhere. The maximal torus is $T \subset \text{SL}_{n+1}$ given by $\mathbb{G}_m^n \subset \text{SL}_{n+1}$

$$(t_1, t_2, \ldots t_n) \mapsto (t_1, t_2, \ldots t_n, (t_1t_2 \ldots t_n)^{-1})$$

For $1 \leq i, j \leq (n+1)$, $i \neq j$, define

$$\phi_{i,j} = a_{i,i}a_{j,j}^{-1}$$

where $a_{i,j}$ are the usual coordinate functions for $M_{(n+1) \times (n+1)}$. Then $\forall t \in T$

$$\text{Ad}(t)E_{i,j} = tE_{i,j}t^{-1} = \phi_{i,j}(t)E_{i,j}$$
\[
\mathfrak{sl}_{n+1} = t \oplus \sum_{i,j} (\mathfrak{sl}_{n+1})_{\phi_{i,j}}
\]

where \((\mathfrak{sl}_{n+1})_{\phi_{i,j}} = \langle E_{i,j} \rangle\).

The characters \(\phi_{i,j} \in X(T)\) satisfy the relations:

1. \(\phi_{i,j} = \phi_{j,i}^{-1}\)
2. \(\phi_{i,j} = \prod_{i \leq m < j} \phi_{m,m+1}\) if \(i < j\).

Let \(E = X(T) \otimes \mathbb{R} \simeq \mathbb{R}^n\). We see that

\[
\alpha_i = \phi_{i,i+1}
\]

is a basis of \(X\); \(a_i\) can viewed as the vector \((0, \ldots, 0, 1, -1, \ldots)\) with 1 in the \(i\)-th position in \(\mathbb{R}^{n+1}\). The \(\phi_{i,j}\) span an \(n\)-dimensional subspace in \(\mathbb{R}^{n+1}\). 
Denote by

\[ \Delta = \{ \alpha_i \} \subset \Phi = \{ \phi_{i,j} \} \]

We have

\[
\dim \text{SL}_{n+1} = (n + 1)^2 - 1 = (n + 1)n + n
\]
\[
= |\Phi| + |\Delta|
\]
\[
= |\Phi| + \text{rk}(T)
\]
The root system of $sl_{n+1}$ is called $A_n$. For example, $A_2$ can be represented as in the picture.
Abstract root system

Let $E/\mathbb{R}$ be a finite vector space.

**Definition**

An (abstract) root system in the real vector space $E$ is a subset $\Phi$ of $E$ satisfying:

1. $\Phi$ is finite, spans $E$, and does not contain $0$. (The elements of $\Phi$ are called roots.)
2. If $\alpha \in \Phi$, the only multiples of $\alpha \in \Phi$ are $\pm \alpha$.
3. If $\alpha \in \Phi$, there exists a reflection $\sigma_{\alpha}$ relative to $\alpha$, which leaves $\Phi$ stable.
4. If then $\alpha, \beta \in \Phi$, $\sigma_{\alpha}(\beta) - \beta$ is an integral multiple of $\alpha$. 
The Weyl group of $\Psi$ is

$$W = W(\Psi) = \langle \sigma_\alpha, \alpha \in \Psi \rangle \subset GL(E).$$

A subset $\Delta \subset \Psi$ is called a basis if $\Delta = \{\alpha_1, \ldots, \alpha_n\}$ is a basis of $E$, relative to which each $\beta \in \Psi$ has a (unique) expression $\beta = \sum_i c_i \alpha_i$, where the $c_i$ are integers of the same sign.

The Weyl group can be identified with $W = N_G(T)/T$, where $N_G(T)/T$ for a maximal split torus $T$ in $G$.

**Example**

For example, for $G = SL_{n+1}$, the Weyl group is $S_n$. The normalizer in $G$ of the diagonal subgroup is the group of monomial matrices, i.e. matrices having exactly one non-zero element for each row and column.
Definition
A Borel $B \subset G$ subgroup is a maximal (closed) connected solvable subgroup.

For example, for $G = \text{GL}_{n,k}$ a Borel subgroup is $T_{n,k}$, the subgroup of upper triangular matrices.

Theorem (Fixed point Theorem)

Let $G$ be a connected solvable group acting on a complete variety $X$. Then $G$ has a fixed point in $X$.

Let $V/k$ be a finite vector space, $\mathcal{F}(V)$ a flag complete flag variety. A connected solvable subgroup $G \subset \text{GL}(V)$ acts on $\mathcal{F}(V)$ and the action has a fixed point. Thus, $G$ stabilizes a complete flag in $V$.
Theorem
- If $B \subset G$ is a Borel subgroup, then $G/B$ is projective.
- All Borel subgroups are conjugate of each other, in particular of the same dimension.

Example
If $V/k$ is a vector space with basis $v_1, \ldots, v_n$, we fix the flag $F = (V_1 \subset V_2 \cdots \subset V_n)$.

$$V_i = \langle v_1, \ldots, v_i \rangle$$

Then $F$ is a point in the flag variety $\mathcal{F}(V)$. If $G = \text{GL}_n(k)$, then

$$\text{Stab}_G(F) = T_{n,k} \quad \text{orbit}(F) = \mathcal{F}(V).$$

Thus, $G/B \simeq \mathcal{F}(V)$ is projective.
Definition of Parabolic subgroups

**Definition**

A closed subgroup $P \subset G$ is **parabolic** if the quotient $G/P$ is projective.

**Theorem**

A closed subgroup $P$ of $G$ is parabolic if and only if it contains a Borel subgroup.

**Proof.**

"⇒" If $P$ is parabolic, $G/P$ is a complete variety, so $B$ fixes a point in $G/P \Rightarrow gBg^{-1} \subset P$;

"⇐" If $B \subset P$, then $G/B \to G/P$ is a surjective map from a complete variety $\Rightarrow G/P$ is complete, hence projective (because $G/P$ is always quasi-projective).
Tits systems

\( G \) group \( \leadsto \) reductive group

\( B, N \) subgroups of \( G \) such that \( G \) is generated by \( N, B \leadsto B \) Borel, \( N \) normalizer of a maximal torus

\( T = B \cap N \) normal in \( N \leadsto T \) maximal torus

\( W = N/T \leadsto W \) Weyl group

\( S \subset W \) a set of involutions generating \( W \). \( \leadsto S = \Delta \)

Definition

We say that \((G, B, N, S)\) is a Tits system if

\( \bullet \) If \( \rho \in S, \sigma \in W, \rho B \sigma \subset B \sigma B \cup B \rho \sigma B \)

\( \bullet \) If \( \rho \in S, \rho B \rho \neq B. \)
Let $(G, B, N, S)$ be a Tits system, $I \subset S$.

**Theorem (Bruhat decomposition)**

1. If $\sigma, \sigma' \in W$, then $B \sigma B = B \sigma' B \iff \sigma = \sigma'$.
2. $G = \bigsqcup_{\sigma \in W} B \sigma B$

In particular, For $I \subset S$, define $W_I = \langle \sigma \mid \sigma \in I \rangle$

$$P_I = BW_I B$$

- $I = \emptyset : P_\emptyset = B$ is the Borel subgroup
- $I = S : P_S = G$
Theorem

1. The $P_I$'s are parabolic subgroups of $G$
2. All the parabolic subgroups of $G$ containing $B$ are of the form $P_I$ for some $I \subseteq S$.
3. If $P_I$ and $P_J$ are conjugate, then $I = J$.

Corollary

Given a Borel subgroup $B \subseteq G$, there exist $2^r$-parabolic subgroups containing $B$ where $r$ is the semisimple rank of $G$.

In particular, in order to determine all the parabolic subgroups, it suffices to find the maximal parabolic subgroups $P_I$ of $I = S \setminus \{i\}$ and take their intersection.
The semisimple rank of $\text{GL}_n$ is $n - 1$. Fix a basis $v_1, \ldots v_n$ and let $V_i = \langle v_1, \ldots v_i \rangle$. Let $F$ be the complete flag $F = (V_1 \subset V_2 \cdot \cdot \cdot \subset V_n)$. Then

**Theorem**

*The parabolic subgroups of $\text{GL}_{n,k}$ containing the standard Borel $B_{n,k}$ are the stabilizers of a subflag of $F$.**

Let $F_I$ with $I \subset \{1, \ldots n - 1\}$ be the flag obtained from $V$ removing the subspaces of indices in $I$. Then $P_I$ is the stabilizer of such flag in the flag variety of corresponding numerical invariants $\mathscr{F}(I)$. All the $P_I's$ look like "upper triangular matrices with blocks on the diagonal".
Parabolic subgroups of $\text{Sp}_{n,k}$

Let $V$ a $k$-vector space with a symplectic pairing $\Psi$. We can find a maximal isotropic basis $v_1, \ldots, v_n$ such that

$$V_i = \langle v_1, \ldots, v_i \rangle \subset \langle v_1, \ldots, v_i \rangle^\perp$$

so that the pairing is given by

$$\Psi = \begin{bmatrix} 0 & \mathbb{1}_n \\ -\mathbb{1}_n & 0 \end{bmatrix}$$

A maximal torus in $T \subset \text{Sp}$ is $\mathbb{G}_m^n \subset \text{Sp}_{n,k}$ as

$$(t_1, t_2, \ldots, t_n) \mapsto \text{diag}(t_1, t_2, \ldots, t_n, t_1^{-1}, t_2^{-1}, \ldots, t_n^{-1})$$

Thus, $\text{rk}(G) = n$. 
Theorem

The parabolic subgroups of $Sp_{n,k}$ are the stabilizers of isotropic flags.

$I \subset \{1,\ldots,n\}$, let $F_I = \{V_1, V_2, \ldots V_j \ldots\}$ be the subflag obtained by omitting $V_i$ with $i \in I$ from the subflag.

$G$ acts on the flag variety $\mathcal{F}(I)$ of numerical invariants $i \notin I$.

The orbit of $F$ is the subvariety of isotropic subspaces and

$$P_I = \text{Stab}_G(F_I).$$

All the $P_I$'s are distinct, and determined by the numerical invariants of the flag.

These are all the parabolics, because the rank of $G$ is $n$. 
Parabolic subgroups of $U(n,m)$

Let $U(n,m)$ be the **real** algebraic group associated to a hermitian form on $\mathbb{C}^{n+m}$ with basis $v_1, v_2, \ldots v_{n+m}$

$$(u, v) = \bar{u}^t \Psi v$$

$$\Psi = \begin{bmatrix} 0 & 0 & 1_m \\ 0 & 1_{n-m} & 0 \\ 1_m & 0 & 0 \end{bmatrix}$$

More precisely, $U(n,m)(\mathbb{R}) = \{ M \in \text{GL}_n(\mathbb{C}) \mid \bar{M}^t \Psi M = \Psi \}$. The maximal torus is isomorphic to $\mathbb{G}_m^m \times U(1)^{n-m}$. via $(t, u) \mapsto (t, u, t^{-1})$. The $\mathbb{R}$-semisimple rank of $G$ is $m$. 
Let \( V_r = \langle e_1, \ldots e_r \rangle \subset V_r^\perp = \langle e_1, e_2 \ldots e_n, e_{n+r+1}, \ldots e_{n+m} \rangle \) for \( 1 \leq r \leq m \).

Let \( F = (V_1, \ldots V_m) \) be the maximal isotropic flag.

**Theorem**

*The parabolic subgroup \( P_I \) of \( U(n,m) \) is the stabilizers of isotropic subflags of \( F \) obtained by removing the \( V_i \)'s for \( i \in I \).*

- The parabolic subgroups \( P_I \) are **real** subgroups, because the flags they stabilize is defined over \( \mathbb{R} \). Since \( m \) is the rank of the maximal split torus over \( \mathbb{R} \), these are all the real parabolic subgroups. In fact \( U(n,m) \) can be defined over \( \mathbb{Q} \) and the \( P_I \)'s are **rational** subgroups.

- \( U(n,m) \cong U(n+m) \cong GL_{n+m},\mathbb{C} \) has rank \( n + m - 1 \), so has \( 2^{n+m-1} \) parabolic subgroups.
Thank you for your attention!